NON-NEWTONIAN FLOW IN A SQUARE DUCT

V. ŠKVOROVÁ

Institute for Hydrodynamics,
Czechoslovak Academy of Sciences, 160 00 Prague 6

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A simple apparatus has been devised to measure pressure drops and volume flow rates under the laminar flow of non-newtonian liquids in a square duct. The measurements were carried out with water solutions of polyacryloamide at three different concentrations displaying a marked non-Newtonian behaviour. A simple two-parameter model due to Rabinowitsch and a three-parameter model due to Sutterby were used to describe the flow properties of the materials used. In case of the former model the experimental data were compared with the values computed from a generalized Rabinowitsch–Mooney equation. In the latter case the data were compared with the results published by Mitsuishi and Aoyagi[1]. These authors applied variational calculus to determine the relationship between flow rate and pressure drop in ducts of non-circular cross section for the Sutterby's model.

EXPERIMENTAL

The apparatus used is essentially a rheometric extruder. The examined liquid is stored in an air-tight tank equipped with a liquid level gauge. Compressed air is brought into the space above the liquid level. Its pressure is controlled by a pressure reducing valve and measured by a U-manometer. The liquid can be discharged from the tank through an attached horizontal brass square duct 0·99 × 98·5 cm. The duct may be easily replaced by a circular tube serving as a rheometer for determination of rheological parameters of the examined liquid. The length of the tube was 98·1 cm and radius 0·49 cm. After removing the seal at the end of the measuring duct the liquid could discharge freely into a vessel at a flow rate assessed from the volume of liquid over the measured time interval.

The apparatus was tested using 99·7% glycerol. The aim of the test was to verify the function of the apparatus and to determine the cross sections of the duct and the tube. Individual points of the Δp–Q dependence were fitted by a straight line the constants of which were determined by the least-square method. The slopes of these straight lines were used to calculate radius of the tube from the Hagen–Poiseuille equation and the size of the square duct from an analogous expression relating flow rate and pressure drop under the laminar flow of newtonian liquids in such ducts.

RESULTS AND DISCUSSION

The measurements were carried out with three water solutions of polyacryloamide of different concentrations. The experimental values of pressure drop, Δp, and flow rate, Q, were used to calculate the consistency variables[2]

$$P = R\,\Delta p/2L\,, \quad V = 4v_0/R = 4Q/\pi R^3\,. \tag{1}$$

Processing of the square duct flow data is conveniently carried out in terms of analogous variables obtained after introducing the hydraulic radius into Eq. (*1*)

$$\bar{P} = a\,\Delta p/4L\,, \quad \bar{V} = 8v_0/a = 8Q/a^3\,. \tag{2}$$

From the relation between the consistency variables in the tube it was apparent that the flow behaviour of neither of the three solutions can be fitted by the power-law model. For easier and more reliable choice of a proper rheological model a rheogram τ_w *versus* γ_w was constructed (*i.e.* the dependence of the shear stress on the shear rate at the wall) for the 4·4% solution of polyacryloamide. For this solution we had collected most $Q - \Delta p$ data. Using this rheogram the analytical model of the flow behaviour due to Rabinowitsch was chosen[2], *i.e.*

$$\gamma = f(\tau) = \varphi_0\tau + \varphi_1\tau^3\,. \tag{3}$$

To test this choice a statistical estimate of the constants of the Rabinowitsch's model was made using the experimental data and the course of the appropriate function was plotted in the rheogram. A comparison of the resulting curve with the experimental data confirmed suitability of the Rabinowitsch's model, Eq. (*3*), for description of the flow behaviour of the material used.

The values of the rheological parameters φ_0 and φ_1 were obtained from the results of measurements with the circular tube. The chosen model is one of those that can be conveniently used to calculate the mean velocity of the laminar flow in a circular tube[3]

$$v_0 = (R/\tau_w^3)\int_0^{\tau_w} \tau^2 f(\tau)\,d\tau\,, \tag{4}$$

where τ_w designates the wall shear stress given by

$$\tau_w = R\Delta p/2L\,. \tag{5}$$

On substituting into Eq. (*4*) from the Rabinowitsch's model, Eq. (*3*), and integrating, a simple relation is obtained between the consistency variables as

$$V = \varphi_0 P + 2/3\varphi_1 P^3\,. \tag{6}$$

The last expression was used to calculate the rheological parameters φ_0 and φ_1

from the experimental data by linear regression analysis for two independent variables. Table I summarizes the computed values of the rheological parameters of the Rabinowitsch's model for the three solutions of polyacryloamide. Fig. 1 is a graphical comparison of the experimental data obtained from measurements in the circular tube with the curves given by Eq. (*6*).

Comparison of Experimental Data with the Results of the Generalized Rabinowitsch–Mooney Equation

A generalization of the Rabinowitsch–Mooney equation[4] leads to an expression analogous to Eq. (*4*) applicable to the flows in non-circular ducts

$$v_0 = (R_h/2k_2)\,\bar{\tau}_w^{-k_1/k_2} \int_0^{\bar{\tau}_w} \bar{\tau}^{(k_1/k_2 - 1)} f(\tau)\,d\tau\,. \qquad (7)$$

In the last equation the radius R is replaced by the hydraulic radius R_h, and $\bar{\tau}_w$ de-

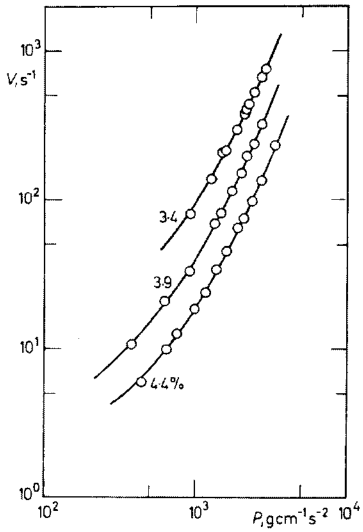

FIG. 1

Experimental Data on the Flow of Water Solutions of Polyacryloamide in Circular Tube

The curves given by Eq. (*6*).

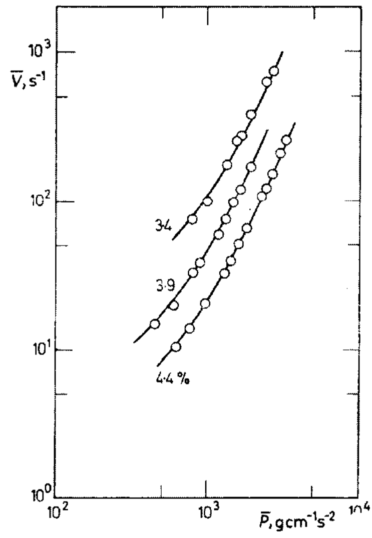

FIG. 2

Experimental Data on the Flow of Water Solutions of Polyacryloamide in Square Duct

The curves given by Eq. (*10*), *i.e.* by the method of the generalized Rabinowitsch–Mooney Eq. (*7*).

signates mean value of the wall shear stress given by

$$\bar{\tau}_w = R_h \, \Delta p / L \, . \tag{8}$$

The parameters k_1 and k_2, appearing in Eq. (*7*), characterize the geometry of cross section of the duct. Their values are assessed from expressions for the mean and maximum velocities of a non-Newtonian liquid in the duct of the given geometry of cross section.

Substituting the Rabinowitsch's model, Eq. (*3*), into Eq. (*7*) simultaneously with the expression for the hydraulic radius and on integration one obtains the following simple relation between the consistency variables defined in Eq. (*2*) for a square duct

$$\bar{V} = \varphi_0[1/(k_1 + k_2)] \, \bar{P} + \varphi_1[1/(k_1 + 3k_2)] \, \bar{P}^3 \, . \tag{9}$$

Substitution of the geometry parameters[4] and the rheological parameters given in Table I leads to the following relations for:

$$\begin{aligned} &3{\cdot}4\% \text{ PAA} \quad \bar{V} = 8{\cdot}02 \, . \, 10^{-2}\bar{P} + 2{\cdot}79 \, . \, 10^{-8}\bar{P}^3 \, , \\ &3{\cdot}9\% \text{ PAA} \quad \bar{V} = 3{\cdot}09 \, . \, 10^{-2}\bar{P} + 1{\cdot}42 \, . \, 10^{-8}\bar{P}^3 \, , \\ &4{\cdot}4\% \text{ PAA} \quad \bar{V} = 1{\cdot}53 \, . \, 10^{-2}\bar{P} + 0{\cdot}561 \, . \, 10^{-8}\bar{P}^3 \, , \end{aligned} \tag{10}$$

A graphical comparison of the experimental results with those calculated according to Eq. (*10*) is shown in Fig. 2. The calculation indicates that the relative deviation of the measured from theoretical values in no case exceeds 8% and remains much lower in the majority of cases.

Comparison of the Experimental Data with the Results of Mitsuishi's Method

Mitsuishi and Aoyagi[1] using variational methods determined the relation between flow rate and pressure drop in non-circular ducts for the three-parameter model due to Sutterby

$$\tau = \eta_0[\text{arcsinh}\,(B\gamma)/B\gamma]^A \, \gamma \, . \tag{11}$$

The results were presented as a plot of $B(4Q/a^2b)$ *versus* (B/η_0). $(a \, \Delta p/2L)$ for various values of the rheological parameter A and geometrical characteristics $E = b/a$ (a and b are respectively the size of the longer and the shorter side of the rectangular duct). Relating Mitsuishi's variables with the consistency variables in Eq. (*2*) we have a plot of $B(\bar{V}/2)$ *versus* $(B/\eta_0) \, 2\bar{P}$. For a square duct $E = 1$.

The rheological parameters of the Sutterby's model, Eq. (*11*), were evaluated from the experimental data using a graphical method[1]. The results are summarized in Table II.

Fig. 3 compares the results of measurements in the square duct with those of the Mitsuishi computational method. The comparison is very favourable and particularly good for the 3·9 and 4·4% solutions of polyacryloamide ($A = 0{\cdot}8$).

TABLE I

Rheological Parameters of the Rabinowitsch's Model for Water Solutions of Polyacryloamide (PAA)

Solution PAA, %	$\varphi_0 \cdot 10^2$ g^{-1} cm s	$\varphi_1 \cdot 10^8$ g^{-3} cm^3 s^5
3·4	7·14	3·66
3·9	2·75	1·86
4·4	1·36	0·735

TABLE II

Rheological Parameters of the Sutterby's Model for Water Solutions of Polyacryloamide

Solution PAA, %	η_0 g cm^{-1} s^{-1}	A	B s
3·4	14·1	1·0	0·0180
3·9	42·5	0·8	0·125
4·4	92·5	0·8	0·257

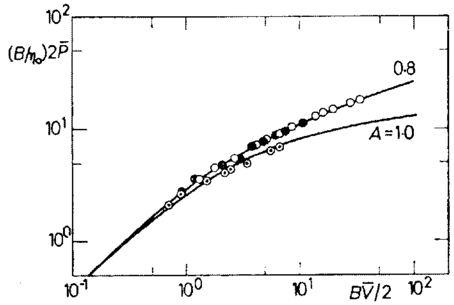

FIG. 3

A Comparison of the Flow Data for Water Solutions of Polyacryloamide in Square Duct with the Results of the Numerical Mitsuishi's Method (solid lines)

⊙ 3·4% PAA, ● 3·9% PAA, ○ 4·4% PAA.

CONCLUSION

The measurements in the circular tube and the processed data confirm suitability of the Rabinowitsch's model, Eq. (*3*), for description of non-Newtonian behaviour of the used solutions of polyacryloamide (Fig. 1). Further it is shown the good agreement of the experimental results of measurement in the square duct with those computed from the generalized Rabinowitsch–Mooney equation[4] (Fig. 2). The suggested computational procedure indicates clearly the ease of applicability of this method to the used rheological model. A comparison of the experimental data with the computed results of the method published in ref.[1] yields also a satisfactory agreement. The comparison is shown in Fig. 3.

LIST OF SYMBOLS

a	size of square duct (cm)
A	rheological parameter of Sutterby's model
B	rheological parameter of Sutterby's model (s)
k_1, k_2	geometrical parameters
L	length of tube and square duct (cm)
p	pressure $(g\,cm^{-1}\,s^{-2})$
P	consistency variable for circular tube defined in Eq. (*1*) $(g\,cm^{-1}\,s^{-2})$
$\bar{P}$	consistency variable for square duct defined in Eq. (*2*) $(g\,cm^{-1}\,s^{-2})$
Q	volume flow rate $(cm^3\,s^{-1})$
R	radius of circular tube (cm)
R_h	hydraulic radius (cm)
v_0	mean velocity $(cm\,s^{-1})$
V	consistency variable for circular tube defined in Eq. (*1*) (s^{-1})
$\bar{V}$	consistency variable for square duct defined in Eq. (*2*) (s^{-1})
γ	shear rate (s^{-1})
γ_w	wall shear rate in circular tube (s^{-1})
η_0	rheological parameter of Sutterby's model $(g\,cm^{-1}\,s^{-1})$
τ	shear stress $(g\,cm^{-1}\,s^{-2})$
τ_w	wall shear stress in circular tube $(g\,cm^{-1}\,s^{-2})$
$\bar{\tau}_w$	mean wall shear stress in non-circular duct defined in Eq. (*8*) $(g\,cm^{-1}\,s^{-2})$
φ_0	rheological parameter of Rabinowitsch's model $(g^{-1}\,cm\,s)$
φ_1	rheological parameter of Rabinowitsch's model $(g^{-3}\,cm^3\,s^5)$

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